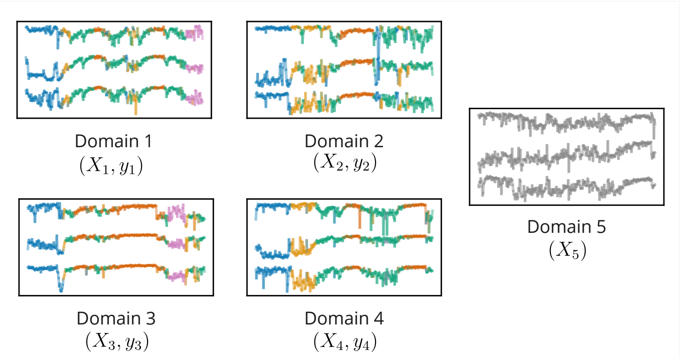
Convolutional monge mapping normalization on sleep data

T. Gnassounou, R. Flamary, A. Gramfort

2023

Multi-source domain adaptation



- K source domains $\{\mathbf{X}_k, y_k\}_{1 \leq k \leq K}$
- One target domain $\mathbf{X}_t$
- **Distribution shift** between domains
- Assumptions: Multivariate Gaussian **stationary** signals

$\Longrightarrow$ How to **reduce the shift** between domains?

Proposition $\rightarrow$ **Mapping to the barycenter** of the domains to cancel their specificities [1].

[1]Montesuma *et al.* , 2021

Monge mapping for Gaussian stationary signals

Let consider Gaussian distributions $\mu_d = \mathcal{N}(\mathbf{m}_d, \mathbf{\Sigma}_d)$ with $d \in \{s, t\}$.
The OT mapping, also called Monge mapping, can be expressed as the following affine function :

$$m(\mathbf{x}) = \mathbf{A}(\mathbf{x} - \mathbf{m}_s) + \mathbf{m}_t, \quad \text{with} \quad \mathbf{A} = \mathbf{\Sigma}_s^{-\frac{1}{2}} \left(\mathbf{\Sigma}_s^{\frac{1}{2}} \mathbf{\Sigma}_t \mathbf{\Sigma}_s^{\frac{1}{2}} \right)^{\frac{1}{2}} \mathbf{\Sigma}_s^{-\frac{1}{2}} = \mathbf{A}^\top .$$

Monge mapping for Gaussian stationary signals

Let consider Gaussian distributions $\mu_d = \mathcal{N}(\mathbf{m}_d, \mathbf{\Sigma}_d)$ with $d \in \{s, t\}$.
The OT mapping, also called Monge mapping, can be expressed as the following affine function :

$$m(\mathbf{x}) = \mathbf{A}(\mathbf{x} - \mathbf{m}_s) + \mathbf{m}_t, \quad \text{with} \quad \mathbf{A} = \mathbf{\Sigma}_s^{-\frac{1}{2}} \left(\mathbf{\Sigma}_s^{\frac{1}{2}} \mathbf{\Sigma}_t \mathbf{\Sigma}_s^{\frac{1}{2}} \right)^{\frac{1}{2}} \mathbf{\Sigma}_s^{-\frac{1}{2}} = \mathbf{A}^\top .$$

The Discrete Fourier Transform (DFT) can diagonalize the circulant matrix

$$\mathbf{\Sigma} = \mathbf{F}\text{diag}(\mathbf{p})\mathbf{F}^* ,$$

with $\mathbf{F}$ and $\mathbf{F}^*$ the Fourier transform operator and its inverse, and $\mathbf{p}$ the Power Spectral Density (PSD) of the signal.

Monge mapping for Gaussian stationary signals

Let consider Gaussian distributions $\mu_d = \mathcal{N}(\mathbf{m}_d, \mathbf{\Sigma}_d)$ with $d \in \{s, t\}$.
The OT mapping, also called Monge mapping, can be expressed as the following affine function :

$$\mathbf{m}(\mathbf{x}) = \mathbf{h} * \mathbf{x} \,, \quad \text{with} \quad \mathbf{h} = \mathbf{F}^* \left(\mathbf{p_t}^{\odot \frac{1}{2}} \odot \mathbf{p_s}^{\odot - \frac{1}{2}} \right) .$$

The Discrete Fourier Transform (DFT) can diagonalize the circulant matrix

$$\mathbf{\Sigma} = \mathbf{F}\text{diag}(\mathbf{p})\mathbf{F}^* \,,$$

with $\mathbf{F}$ and $\mathbf{F}^*$ the Fourier transform operator and its inverse, and $\mathbf{p}$ the Power Spectral Density (PSD) of the signal.

Wasserstein barycenter between Gaussian stationary signals

Considering multiple Gaussian distributions μ_k. The barycenter $\bar{\mu}$ is expressed as

$$\bar{\mu} = \arg\min_{\mu} \frac{1}{K} \sum_{k=1}^{K} \mathcal{W}_2^2(\mu, \mu_k) \ . \tag{1}$$

The barycenter is still a Gaussian distribution $\bar{\mu} = \mathcal{N}(\bar{\mathbf{m}}, \bar{\mathbf{\Sigma}})$.

$\implies$ **No closed-form** for computing the covariance $\bar{\mathbf{\Sigma}}$.

One uses the following optimality condition from [a]

$$\bar{\mathbf{\Sigma}} = \frac{1}{K} \sum_{k=1}^{K} \left(\bar{\mathbf{\Sigma}}^{\frac{1}{2}} \mathbf{\Sigma}_k \bar{\mathbf{\Sigma}}^{\frac{1}{2}} \right)^{\frac{1}{2}} \ ,$$

[a] Agueh *et al.* Barycenters in the Wasserstein Space, 2011, SIAM

Wasserstein barycenter between Gaussian stationary signals

Considering multiple Gaussian distributions μ_k. The barycenter $\bar{\mu}$ is expressed as

$$\bar{\mu} = \arg\min_{\mu} \frac{1}{K} \sum_{k=1}^{K} \mathcal{W}_2^2(\mu, \mu_k) \ . \tag{1}$$

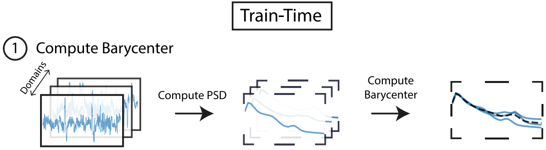
The barycenter is still a Gaussian distribution $\bar{\mu} = \mathcal{N}(\bar{\mathbf{m}}, \bar{\boldsymbol{\Sigma}})$.

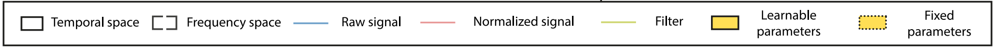
Lemma from Gnassounou *et al.*

Consider K centered stationary Gaussian signals of PSD $\mathbf{p}_k$ with $k \in [K]$, the Wasserstein barycenter of the K signals is a centered stationary Gaussian signal of PSD $\bar{\mathbf{p}}$ with:

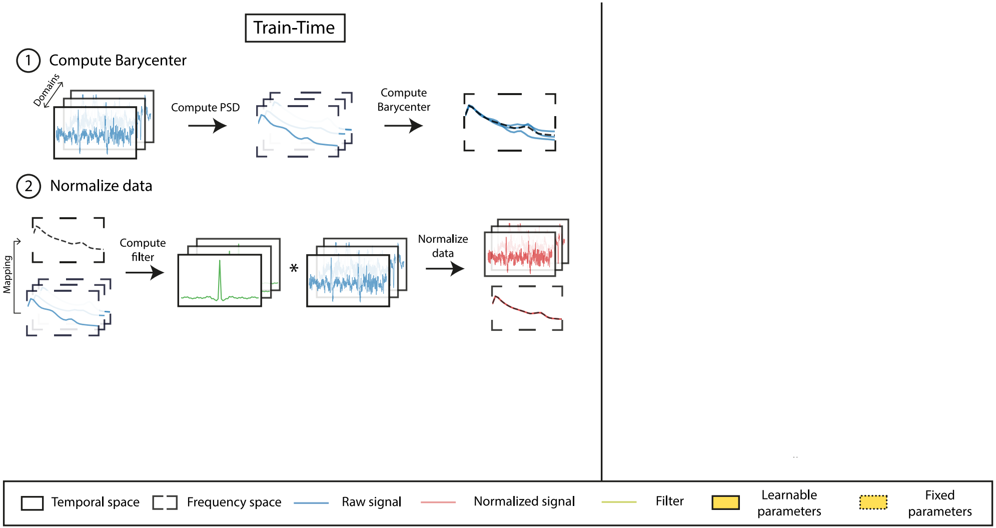
$$\bar{\mathbf{p}} = \left(\frac{1}{K} \sum_{k=1}^{K} \mathbf{p}_k^{\odot \frac{1}{2}} \right)^{\odot 2} \ . \tag{2}$$

Convolutional Monge Mapping Normalization (CMMN)

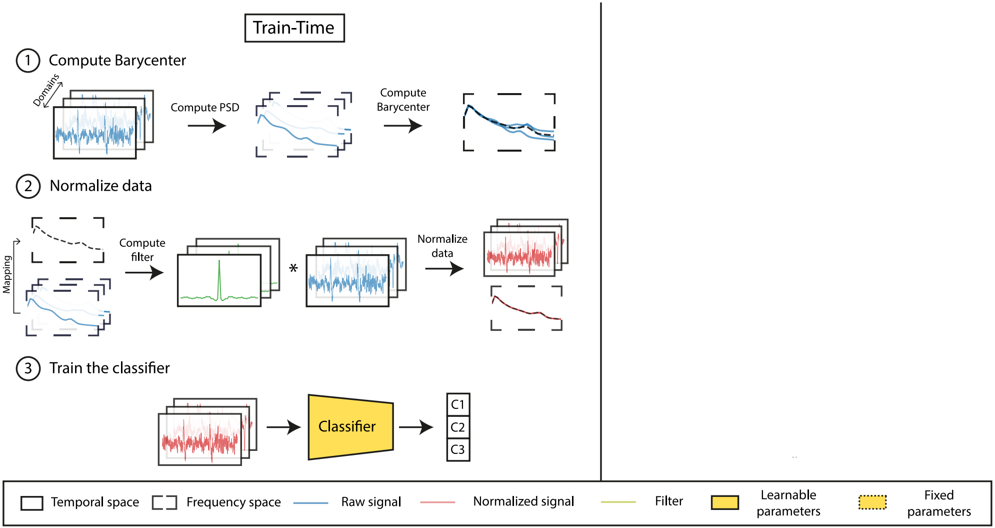




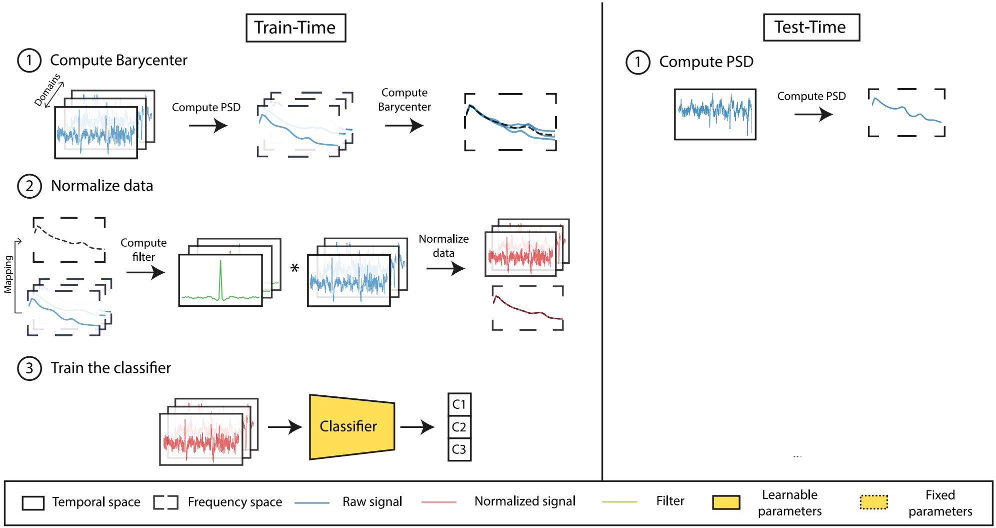
Convolutional Monge Mapping Normalization (CMMN)



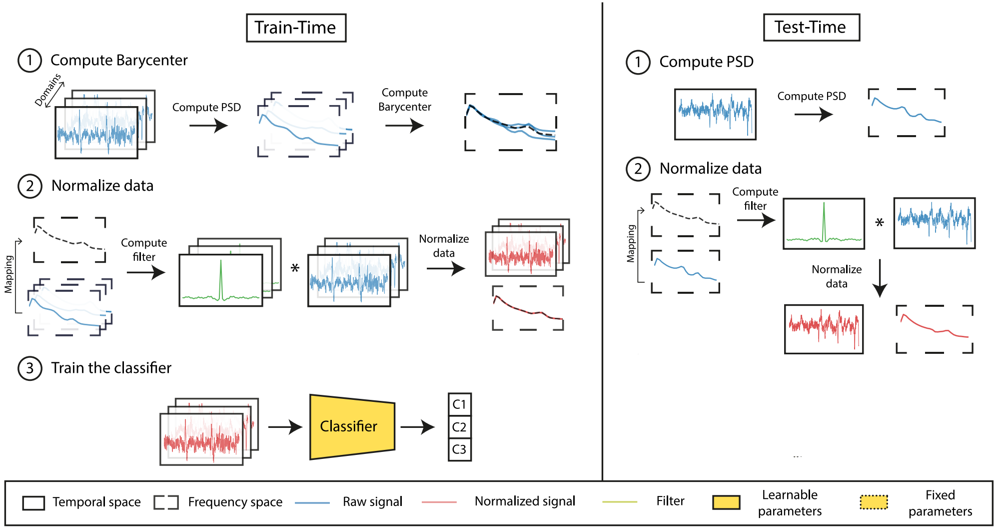
Convolutional Monge Mapping Normalization (CMMN)



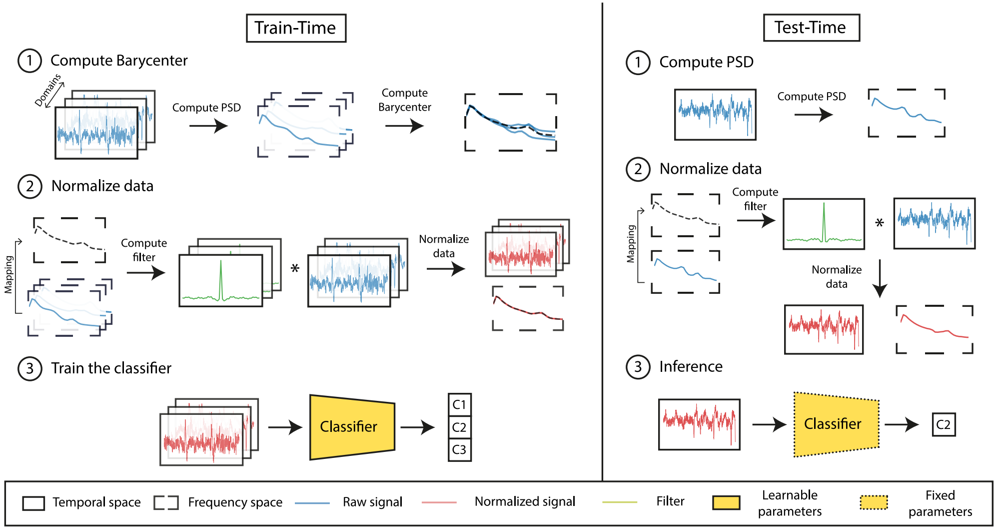
Convolutional Monge Mapping Normalization (CMMN)



Convolutional Monge Mapping Normalization (CMMN)



Convolutional Monge Mapping Normalization (CMMN)



Conclusion

Come see the poster for the results on sleep staging!